C. Structure Groups of Fiber Bundles

given a fiber bundle
$$\begin{tikzcd} F \arrow[r] & E \arrow[d, "p"] \\ & B \end{tikzcd}$$

we have local trivializations

$U \subset B$ open and a diffeomorphism

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\phi"] \arrow[dr, "p"'] & \arrow[d, phantom, "\circ"] & U \times F \arrow[dl, "p_1"] \\ & U & \end{tikzcd}$$

and for two local trivializations (U_1, ϕ_1) and (U_2, ϕ_2) we have

$$\begin{tikzcd} (U_1 \cap U_2) \times F \arrow[dr, "p_1"'] & p^{-1}(U_1 \cap U_2) \arrow[l, "\phi_1"'] \arrow[r, "\phi_2"] \arrow[d, "p"] & (U_1 \cap U_2) \times F \arrow[dl, "p_1"] \\ & U_1 \cap U_2 & \end{tikzcd}$$

$$\phi_2 \circ \phi_1^{-1} : (U_1 \times U_2) \times F \longrightarrow (U_1 \times U_2) \times F$$
$$(x, y) \longmapsto (x, \tau_{21}(x)(y))$$

group of homeomorphisms

where $\tau_{21} : U_1 \cap U_2 \longrightarrow \mathrm{Homeo}(F)$

transition function or
clutching function

note: if $\{(U_\alpha, \phi_\alpha)\}$ a collection of local trivializations
such that $B = \bigcup U_\alpha$
then the transition maps satisfy

$$\boxed{\begin{aligned} &\tau_{\alpha\alpha}(x) = \mathrm{Id}_F \\ &\tau_{\beta\alpha}(x) = (\tau_{\alpha\beta}(x))^{-1} \\ &\tau_{\gamma\alpha}(x) = \tau_{\gamma\beta}(x) \circ \tau_{\beta\alpha}(x) \end{aligned}} \quad (*)$$

exercise: Show that if $\{U_\alpha\}$ is a cover of B by open sets and $\tau_{\alpha\beta}: U_\alpha \cap U_\beta \to \mathrm{Homeo}(F)$ are maps satisfying (*) then $\exists$ a bundle E over that realizes this data

Hint: let $E = \left(\coprod_\alpha (U_\alpha \times F)\right)/\sim$ where $(x,y) \in U_\alpha \times F \sim (x',y') \in U_\beta \times F$ iff $\tau_{\beta\alpha}(x)(y) = y'$ and $x = x'$

there is an obvious projection to $E \to B$

exercise: Find an open cover and transition functions for

$$\begin{array}{ccc} S^1 \to S^{2n+1} & \mathbb{R}^n \to TS^n & \mathbb{R}^{2n} \to T\mathbb{C}P^n \\ \downarrow & \downarrow & \downarrow \\ \mathbb{C}P^n & S^n & \mathbb{C}P^n \end{array}$$

Suppose $G \subset \mathrm{Homeo}(F)$ is a sub topological group (we will only consider closed subgroups)

if $\begin{array}{c} F \to E \\ \downarrow p \\ B \end{array}$ has a collection of transition functions

$$\tau_{\alpha\beta}: U_\alpha \cap U_\beta \to G$$

we say E has structure group G

if the trasition functions are not in G but can be homotoped to be in G via a homotopy that always satisfies (*) then we say the structure group reduces to G

note: If G preserves some structure on the F, then the fibers of $p: E \to B$ have this structure

examples:

1) if $F = \mathbb{R}^n$ and $G = GL(n;\mathbb{R}) \subset \mathrm{Homeo}(\mathbb{R}^n)$ then each fiber has a linear structure

ie. E is a vector bundle

2) if $F = \mathbb{R}^n$ and $G = GL^+(n;\mathbb{R})$, then E is an oriented vector bundle

3) if $F = \mathbb{R}^n$ and $G = O(n)$, then E is a vector bundle with a metric

note: $O(n) \hookrightarrow GL(n;\mathbb{R})$ is a homotopy equivalence $\Rightarrow$ all bundles have metrics!

4) if $F = \mathbb{R}^{2n}$, then

$G = GL(n;\mathbb{C}) \Leftrightarrow E$ has complex structure

$G = U(n) \Leftrightarrow E$ has a Hermitian structure

5) if $F = \mathbb{R}^n$ and $G = GL(k) \times GL(n-k) \subset GL(n)$

$$(A, B) \longmapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

then E has structure group G

$\Leftrightarrow$

$$E \cong E_1 \oplus E_2$$

for E_1 an $\mathbb{R}^k$-bundle and E_1 an $\mathbb{R}^{n-k}$-bundle

similarly if $G = GL(n-k) \subset GL(n)$ then

E has structure group G

$\Longleftrightarrow$

$E = E' \oplus \mathbb{R}^k$ with E' an $\mathbb{R}^{n-k}$-bundle

So when can you reduce the structure group?

if G is a Lie group (or topological group) then a bundle

$$\begin{array}{l} G \to P \\ \quad \downarrow p \\ \quad M \end{array}$$

is a principal G-bundle if

$\exists$ a smooth (or continuous) right G-action

$$P \times G \to P$$

such that

1) action preserves fibers

i.e. $y \in p^{-1}(x) \Rightarrow y \cdot g \in p^{-1}(x) \; \forall g, x, y$

2) G acts freely and transitively on $p^{-1}(x) \; \forall x$

Remark: can also define a smooth principal G-bundle as a smooth manifold P with a smooth right G-action $P \times G \to P$ that is free and proper

if for map $P \times G \to P \times P$
$(p, g) \mapsto (p \cdot g, p)$
preimage of compact is compact

examples:

1) if $\begin{array}{c} F \to E \\ \downarrow \\ M \end{array}$ is a bundle with structure group G

then there is a cover of M by loc. triv. $\{(U_\alpha, \phi_\alpha)\}$

with transition functions $\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to G$

we can construct a principal bundle as follows

$$P_E = \coprod_\alpha U_\alpha \times G /\sim$$

where $(x, g) \in U_\alpha \times G \sim (x', g') \in U_\beta \times G$

$$\Leftrightarrow$$

$$g' = \underbrace{\tau_{\beta\alpha}(x)}_{\in G} \cdot g \quad x = x'$$

exercise: check P_E is a principal G-bundle

if E is a vector bundle then P_E is a principal $GL(n;\mathbb{R})$-bundle. It is called the frame bundle because you can think of the fibers of P_E as frames for the fibers of E

exercise: think through this

we denote this bundle $\mathcal{F}(E)$

note: $O(n) \simeq GL(n;\mathbb{R})$ so we could look at the orthonormal frame bundle with fiber $O(n)$, still call it $\mathcal{F}(E)$

2) $\begin{array}{ccc} S^1 \to & S^{2n+1} \\ & \downarrow \\ & \mathbb{C}P^n \end{array}$ is a principal S^1-bundle

3) regular covering spaces of a manifold are principal bundles

exercise: Check this. What are the fibers? can an irregular cover be a principal bundle?

exercise:

1) Show a principal G-bundle is trivial $\Leftrightarrow$ it has a section
2) If E is a vector bundle, then a section of E is the same as a $GL(n,\mathbb{R})$-equivariant map
$$v: \mathcal{F}(E) \to \mathbb{R}^n$$
(ie. $v(y \cdot g) = g^{-1} v(y)$)

Hint: given $s: M \to E$ then for each $y \in \mathcal{F}(E)$ let $v(y) = s(p(y))$ expressed in frame y

projection $p: \mathcal{F}(E) \to M$

Construction:

Given $\begin{array}{c} P \\ \downarrow \\ M \end{array}$ a principal G-bundle, and

$\rho: G \to G'$ a homomorphism (of Lie groups) where $G' \subset \mathrm{Homeo}(F)$

then we can construct an F-bundle with structure group G'

$$P \times_\rho F = P \times F \big/ (p \cdot g, f) \sim (p, \rho(g) \cdot f)$$

exercise:

1) Describe $P \times_\rho F$ using local trivializations

2) if $F = G'$ then $P \times_\rho G'$ is a principal G'-bundle

3) if E is a vector bundle, then

$$E \cong \mathcal{F}(E) \times_\rho \mathbb{R}^n$$

where $\rho = id_{GL(n,\mathbb{R})}$

4) recall $GL(n,\mathbb{R})$ acts on $(\mathbb{R}^n)^*$ in a natural way

e.g. given $A \in GL(n,\mathbb{R})$ we have

$$A : \mathbb{R}^n \to \mathbb{R}^n$$

so

$$A^* : (\mathbb{R}^n)^* \to (\mathbb{R}^n)^*$$

$$(\ell : \mathbb{R}^n \to \mathbb{R}) \mapsto \ell \circ A : \mathbb{R}^n \to \mathbb{R}$$

call this $\rho^* : GL(n;\mathbb{R}) \to GL((\mathbb{R}^n)^*)$

check $T^*M \cong \mathcal{F}(TM) \times_{\rho^*} (\mathbb{R}^n)^*$

5) Similarly $GL(n;\mathbb{R})$ acts on $\Lambda^k(\mathbb{R}^n)^*$ in a

natural way $GL(n;\mathbb{R}) \xrightarrow{\rho_k} GL(\Lambda^k(\mathbb{R}^n)^*)$

check $\Lambda^k(T^*M) \cong \mathcal{F}(TM) \times_{\rho_k} \Lambda^k(\mathbb{R}^n)^*$

now given a principal G-bundle $\begin{array}{c} P \\ \downarrow \\ M \end{array}$ and a subgroup $H < G$

if $\exists$ a principal H-bundle $P_H \subset P$ then one can check

$$P_H \times_H G \cong P : (f,g) \longmapsto f \cdot g$$

H acts on G by multiplication

is a bundle isomorphism

this isomorphism shows that the transition functions for P could be chosen to have image in H

so the structure group of P reduces to H

note: if the structure group of $\mathcal{F}(E)$ reduces from $GL(n;\mathbb{R})$ to H, then so does the structure group of E:

$$E \cong \mathcal{F}(E) \times_H \mathbb{R}^n$$

$H < GL(n;\mathbb{R})$ acts on $\mathbb{R}^n$

so we have turned questions about the structure group of a vector bundle E into questions about the structure group of principal bundles

Even more! classifying vector bundles with structure group H is the same as classifying principal bundles with structure group H.

now given a principal G-bundle $\begin{matrix} P \\ \downarrow \\ M \end{matrix}$ and a subgroup $H < G$ we get the bundle

$$\begin{matrix} P/H \\ \downarrow \\ M \end{matrix} \quad \text{with fiber } G/H$$

not nec. a group!

lemma 12:

> let P be a principal G-bundle and $H < G$
> reductions of the structure group of P to H are in one-to-one correspondence with sections of P/H

Proof:

($\Rightarrow$) given a reduction we have

$$\begin{tikzcd} P_H \arrow[r, hook] \arrow[dr] & P \arrow[d] \\ & M \end{tikzcd}$$

and so

$$\begin{tikzcd} P_H/H \arrow[r, hook] \arrow[dr, "\cong"'] & P/H \arrow[d] \\ & M \arrow[u, bend left, "\text{section}"] \end{tikzcd}$$

($\Leftarrow$) note $P \xrightarrow{\pi} P/H$ is a principal H-bundle

if $s: M \to P/H$ is a section, then

$$\bigcup_{x \in M} \pi^{-1}(s(x)) \subset P$$ is a principal

H-bundle

<u>example</u>: since $GL(n;\mathbb{R})/O(n)$ is contractible and bundles with contractible fibers always have a section (<u>exercise</u>: check this! or better see next section) we see $\mathcal{F}(E)/O(n)$ has a section and so all vector bundles have metrics!

So how can we tell if P/H has sections?

<u>answer:</u> See next section. Obstruction Theory